Optimization, Written Assignment #2

April 16, 2008

All numbered exercises are from Boyd and Vandenberghe.

1. Convexity
   (a) Prove that if f is convex and differentiable then x is a global minimum of f iff $\triangledown f(x) = 0$
   (b) We say that x is a "local minimum" of a f if there exists some $\epsilon > 0$ such that $f(x) \leq f(y)$ for all y such that $\|x - y\|_2 < \epsilon$. Prove that every local minimum of a convex (not-necessarily differentiable) function is also a global minimum
   (c) Can a convex function have multiple global minima (give an example)? Prove that the set of global minima is convex
2. In this problem we will consider gradient descent with predetermined step sizes
   (a) Show that a constant step size of $t = 1$ can lead to a sequence of iterates $x^{(k)}$ that do not converge to the minimum of f, even for convex f (hint: it is enough to consider optimization over $\mathbb{R}$)
   (b) Show that for *any* constant step size, there is a convex function f for which this step size can lead to a sequence of iterates that do not converge to the minimum of f
   (c) Optional: Now consider a pre-determined sequence of step sizes $t^{(k)}$ such that $\lim_k t^{(k)} = 0$, with $\sum_{k=0}^{\infty} t^{(k)} = \infty$. Assume f is convex and its gradient is Lipschitz continuous, so there exists a finite constant L such that:

   $$\|\triangledown f(x) - \triangledown f(y)\|_2 \quad \leq \quad L\|x - y\|_2$$

   for all x, y in the domain of f. Show that for any such sequence $t^{(k)}$, any such function f, and any starting point $x^{(0)}$, we will have that $\lim_k f\left(x^{(k)}\right) = p^*$
   (d) Optional: Why is the requirement that $\sum_{k=0}^{\infty} t^{(k)}$ necessary?
3. Problem [9.5]
4. Optional: Problem [9.8]
5. Optional: Problem [9.18] (note that Boyd and Vandenberghe use a slightly different version of self-concordance than that given in class: they use a fixed $r = 0.5$)
6. Optional: Problem [9.19]
7. Problem [9.21]
8. Problem [9.23]
   (a) Optional
   (b) Required!
   (c) Optional